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An Inviscid Regularization of Hyperbolic Conservation Laws

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Abstract This article examines the utilization of a spatial averaging technique that applies filtering to the nonlinear terms of the partial differential equations as an inviscid shock-regularization of hyperbolic conservation laws. A central motivation is to promote a recently developed filtering technique, rather than viscous regularization, as an alternative to the simulation of shocks and turbulence for inviscid flows. On the other hand, the results generalize and unify previous mathematical and numerical analysis of the method applied to the one-dimensional Burgers' and Euler equations. This article primarily concerns the mathematical analysis of this filtering technique and examines two fundamental issues. The first is the global existence and uniqueness of classical solutions for the regularization under the more general setting of quasilinear, symmetric hyperbolic systems in higher dimensions. The second issue examines one-dimensional scalar conservation laws and shows that the inviscid regularization method captures the unique entropy or physically relevant solution of the original, non-averaged problem as filtering vanishes.

Keywords conservation laws · filter · Helmholtz · regularization · spatial averaging

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1 Introduction

The governing equations for inviscid fluid flow—the Euler equations—although well-known, still pose great challenges in their computation. This difficulty is attributed to the nonlinear convective term in the equations and obtaining a deeper understanding of this model centers on generating novel methods for studying two important features: shock formation along with its regularization and turbulence. More specifically, the culprits are the nonlinear terms which are responsible for the continuous generation of energy transfered to ever smaller scales resulting in a very high resolution problem. It is customary in many numerical simulations to introduce artificial viscosity into the Euler equations to mitigate this energy cascade and resolve the smaller scales thereby circumventing the computational difficulties. Alternatively, the regularization technique studied in this article implements a Leray-type filtering or spatial averaging of nonlinear terms in order to achieve the same goal, however, the focus here centers on scalar conservation laws as a regularization of shocks. When applied to scalar conservation laws, the regularization can be regarded as an averaging of the characteristics and it is a direct extension of the technique proposed in [24] in which averaging of the convective term is introduced in the Burgers' equation via the Helmholtz filter:

$$\begin{cases} \partial_t u^\alpha + \overline{u^\alpha} u^\alpha_x = 0, \\ \overline{u^\alpha} = g^\alpha * u^\alpha, \\ g^\alpha(x) = \frac{1}{2\alpha} e^{-|x|/\alpha}, \end{cases} \tag{1.1}$$

where $*$ denotes the convolution product. Informally speaking, the technique in this article regularizes scalar conservation laws,

$$\partial_t u + f(u)_x = 0,$$

into the form

$$\partial_t u + \overline{\frac{\partial f}{\partial u}} u_x = 0,$$

where the filtered quantity is the convolution product of that term with respect to some averaging kernel as was done in (1.1). This approach is not without proper physical development in the sense that we do not just apply filtering casually. Interestingly enough, this implementation of filtering to the nonlinear term can be interpreted as an extension of the observable divergence method introduced in [23,26]. In fact, the two regularization techniques are equivalent with respect the Helmholtz filter and for quadratic polynomial fluxes, and the reader is referred to [23] for more details on the observable divergence method including the physical derivation of the filtered conservation laws from basic principles.

Let us briefly motivate why we strongly believe that this approach may possess favorable attributes and advantages over viscous regularization; we demonstrate this by considering the simplest case of a nonlinear transport equation that exhibit shocks: the initial value problem to Burgers' equation,

$$\begin{cases} \partial_t u^\epsilon + u^\epsilon u^\epsilon_x = \epsilon u^\epsilon_{xx}, \\ u^\epsilon(x,0) = u_0(x). \end{cases} \tag{1.2}$$

In the presence of viscosity ($\epsilon > 0$), the equation is of the parabolic type and the regularizing effects of viscous perturbations are well-known. This may be thought of as a global regularization since the solution depends on the initial condition i.e. information propagates at 'infinite' speed. The filtered equation (1.1) is modified locally in the sense that the characteristics are mollified so that the equation is still of the hyperbolic type and information propagates along characteristics at finite speed. It is reasonable to suggest that this mollification of the characteristics may preserve important features of the Burgers' equation. Moreover, there is still much debate on the limit of vanishing viscosity for the full Euler equations. Thus, this notion of applying an averaging process instead as a potential alternative to alleviating the computational issues in the simulation of inviscid flows has garnered some recent attention [3, 7, 16, 24, 27, 28, 29]. Although the previous remarks regarding the potential effectiveness of this averaging framework seems reasonable, a rigorous mathematical analysis of the technique is required and will be the main objective of this paper. More precisely, we address the following theoretical aspects for the regularization technique applied to scalar conservation laws: determine the suitable conditions in order to

- establish the global existence and uniqueness of classical solutions,
- establish the convergence to weak solutions, and
- verify that the limiting weak solution is entropy admissible.

Let us remark on some past results for (1.1). In [3], the global well-posedness of regular solutions and the convergence to weak solutions as α vanishes was shown for (1.1) including supporting numerical results suggesting that the entropy solution is captured. The authors in [27] examined the multi-dimensional version of this regularized Burgers' equation and developed similar results such as global existence and uniqueness of classical solutions. They also studied the conserved quantities, traveling wave solutions, spectral energy decay properties of this regularized model and numerically compared its smoothing features with the case when viscosity is present. The convergence as filtering vanishes to the entropy solution was partially resolved in [28] for C^1 bell-shaped initial conditions. Here the approach for showing convergence to the entropy solution uses the Lax admissibility condition rather than the Krŭzkov entropy condition, therefore the methods used are vastly different than the ones presented here. So it is interesting to note that one can approach this question of uniqueness of physical solutions using different admissibility conditions. Another work examining convergence to the unique solution of an initial-boundary value problem as viscosity vanishes using non-Krŭzkov conditions can be found in [1].

We should mention, though, that such an idea is not entirely new. Leray proposed using a filtered convective velocity to study the Navier-Stokes equations [18]. This influenced the investigation of the Leray-α models of turbulence [7, 19], and the Lagrangian Averaged Navier-Stokes-α (LANS-α) implements similar filtering to successfully model turbulent incompressible flows [6, 11, 14, 13, 21, 25, 31]. On the other hand, we must stress that we are adopting filtering in a slightly different approach. From a computational standpoint, this method can be implemented in the modeling of inviscid fluid flows. Since both turbulence and shock formation are nonlinear features, we believe that filtering may lead to a method that can model both nonlinear features in one single comprehensive technique. In this manuscript, this filtering addresses the first issue—determining its potential as a shock regularization technique. In the time of this writing, we are not aware of any other work in the literature which considers this approach in the general setting of hyperbolic conservation laws.

As a regularization of shocks, it is quite natural to seek an understanding of the 'smoothing' effects that such an averaging procedure will have on these evolution equations. For instance, recall that it is well-known that hyperbolic conservation laws generally exhibit finite-time gradient blow-up of classical solutions even if the initial data is smooth. Consequently, this leads us to seek global-in-time solutions in a broader class of discontinuous functions and to introduce so-called entropy conditions to identify physically relevant solutions. In fact, such entropy conditions may allow for the uniqueness and stability of these entropy solutions as illustrated by the Krŭzkov estimates for scalar conservation laws [4,8,12]. Hence, in this manuscript, we apply this technique of spatial averaging or filtering of the non-linear terms in the partial differential equations (PDEs) with the purpose of establishing the analytical properties settling the debate on its validity as a shock-regularization technique. More precisely, we verify two very fundamental properties. First, the regularized PDEs should possess the global existence and uniqueness of classical solutions. Second, and perhaps the more difficult property to prove, is the convergence of the regularized solutions to the physically relevant entropy solution of the original problem as filtering vanishes. We should mention as well that the global existence theory developed here unifies and further generalizes the results obtained in [27,29].

This manuscript is organized as follows. Section 2 introduces our regularization technique along with the class of averaging kernels or filters that will be considered. In section 3, we extend our regularization technique to more general quasilinear, symmetric, hyperbolic systems

$$\partial_t u + \sum_i^n A_i(x,t,u)u_{x_i} = h(x,t,u) \ \text{ in } \mathbb{R}^n \times (0,T). \tag{1.3}$$

Here we provide a theorem addressing the sufficient conditions that guarantee the global well-posedness of classical solutions for the Cauchy problem to (1.3) with spatial averaging. We remark that in comparison with the previous references, our proof moves away from the use of the method of characteristics, and in turn allows us to obtain global well-posedness for a larger class of hyperbolic problems in higher dimensions. Nonetheless, the establishment of the energy estimates illustrates the mechanism in the filtering which prevents the blowup of solutions. In section 4, we describe the conditions required to show that the sequence of regularized solutions to non-homogeneous conservation laws (with source terms) converges to a weak solution of the original system. Furthermore, in the absence of source terms, we prove that this limit is entropy admissible. We study the case with quadratic flux and the Helmholtz filter as a motivating example. Here we exploit the special structure inherent in this case. Consequently, it demonstrates why the past references achieved the aforementioned results limited to this simple case. The final section provides some concluding remarks and suggestions for possible directions and generalizations to this averaging method. This includes several proposed extensions such as the observable divergence method which appear to be computationally favorable in the study of the 1d Euler equations. Other important considerations include initial-boundary value problems for this inviscid regularization.

2 Filters and the method of spatial averaging

Let G be a given real-valued function in $W^{1,1}(\mathbb{R}^n)$ that is positive, symmetric, and monotonically decreasing with unit length in $L^1(\mathbb{R}^n)$. The function G will be referred to as a *filter* or *averaging kernel*. In addition, we shall prescribe a fixed parameter, $\alpha > 0$, to the filter such that

$$G^\alpha = \frac{1}{\alpha^n} G\left(\frac{\mathbf{x}}{\alpha}\right).$$

This parameter α acts as a scaling of the filter and controls the level of averaging and allows us to interpret the $\alpha \to 0$ limit to be when filtering vanishes. In fact, G^α converges to the Dirac delta distribution as filtering vanishes. Table 2.1 summarizes the properties for the filters that we shall consider.

Properties	Mathematical Expression
Normalized	$\|G\|_{L^1(\mathbb{R}^n)} = 1$
Non-negative	$G(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$
Symmetric	$\|\mathbf{x}_1\| = \|\mathbf{x}_2\| \Rightarrow \|G(\mathbf{x}_1)\| = \|G(\mathbf{x}_2)\|$
Non-increasing	$\|\mathbf{x}_1\| \leq \|\mathbf{x}_2\| \Rightarrow \|G(\mathbf{x}_1)\| \geq \|G(\mathbf{x}_2)\|$

Table 2.1 The properties of the averaging kernels

Remark: One example of a commonly applied filter is the Helmholtz filter, $f = \overline{f} - \overline{f}_{xx}$, corresponding to the averaging kernel $G = \frac{1}{2}e^{-|x|}$. This filter was extended then studied in [28] to filters satisfying the form

$$f = \left(1 + \sum_{j=1}^{N_0} (-1)^j C_j \frac{\partial^{2j}}{\partial x^{2j}}\right) \overline{f}$$

for some finite natural number N_0 and constants C_j.

In section 3, we consider the Cauchy problem

$$\begin{cases} \partial_t u + \displaystyle\sum_{i=1}^{n} A_i(x,t,u) u_{x_i} = h(x,t,u) & \text{in } U_T = \mathbb{R}^n \times (0,T), \\ \qquad\qquad u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Here, the A_i's are symmetric $N \times N$ matrix-valued maps defined on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^N$, u and h are N vector-valued maps on $\mathbb{R}^n \times [0,T]$ and $\mathbb{R}^n \times [0,T] \times \mathbb{R}^N$, respectively. Let H^k and $W^{k,p}$ denote the Sobolev spaces $H^k(\mathbb{R}^n, \mathbb{R}^N)$ and $W^{k,p}(\mathbb{R}^n, \mathbb{R}^N)$ equipped with their usual norms $\|\cdot\|_k := \|\cdot\|_{H^k}$ and $\|\cdot\|_{k,p} := \|\cdot\|_{W^{k,p}}$, respectively.
Given a filter G, our regularization technique modifies (2.1) into

$$\begin{cases} \partial_t u^\alpha + \displaystyle\sum_{i=1}^{n} \overline{A_i(x,t,u^\alpha)} u^\alpha_{x_i} = h(x,t,u^\alpha) & \text{in } U_T, \\ \qquad\qquad u^\alpha(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{2.2}$$

where the bar represents the convolution product taken with respect to the filter G in the x-variables. More precisely,

$$\begin{aligned} \overline{A}_i &= \left[\ \overline{a_i^{jl}}\ \right] = \left[G^\alpha * a_i^{jl}(x,t,u^\alpha)\right] \\ &= \left[\int_{\mathbb{R}^n} G^\alpha(x-y) a_i^{jl}(y,t,u^\alpha(y,t))\, dy\right]. \end{aligned}$$

In section 4, we study the corresponding scalar conservation law

$$\begin{cases} \partial_t u + f(u)_x = h(x,t,u) & \text{in } \mathbb{R} \times (0,T), \\ \qquad u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

which is regularized into its filtered quasilinear counterpart

$$\begin{cases} \partial_t u^\alpha + \overline{a(u^\alpha)} u^\alpha_x = h(x,t,u^\alpha) & \text{in } \mathbb{R} \times (0,T), \\ \qquad u^\alpha(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $a = \frac{\partial f}{\partial u}$. For the sake of brevity, we sometimes omit the α–superscript in the averaged equations above, however, it should always be assumed that a corresponding α is given.

3 Global existence for quasilinear, symmetric hyperbolic systems

This section provides the sufficient conditions which guarantee the existence and uniqueness of classical solutions for the regularized problem (2.2). We incorporate the usual methods for quasilinear, symmetric hyperbolic systems and illustrate through the Sobolev energy estimates the regularizing effects of averaging the nonlinear terms of the equations through a low pass filter such as the Helmholtz filter. Although the main idea is standard, we provide a detailed overview of the proof for the sake of completeness. Moreover, we briefly state some basic existence theory and H^k-energy estimates for linear systems under (C-1) conditions; but the reader is referred to [5, 10,15,20,22,30] for further details.

Quasilinear symmetric hyperbolic systems

Define $B_R \subset H^k$ to be the closed ball with radius R. The following conditions are assumed and will be referred to as condition (C-2).

(a) $k > 1 + n/2$ and $u_0 \in H^k$.
(b) For given $u \in H^k$, $A_i(x,t,u(x,t))$ and $h(x,t,u(x,t))$ are H^k-functions that satisfy (C-1) i.e.
  - A_i are symmetric.
  - $t \mapsto A_i(t) \doteq A_i(\cdot,t)$ is of class $C([0,T], H^k(\mathbb{R}^n, \mathbb{R}^{N^2}))$.
  - $t \mapsto h(t) \doteq h(\cdot,t)$ is of class $C([0,T], H^k(\mathbb{R}^n, \mathbb{R}^{N}))$.

(c) The maps $u \in B_R \mapsto A_i(x,t,u)$ and $u \in B_R \mapsto h(x,t,u)$ are bounded (maps bounded sets to bounded sets in H^k) and are C^1 maps with bounded derivatives.

The main global existence result is given in the follow theorem.

Theorem 3.1 *For each $\alpha > 0$, the initial value problem (2.2) under (C-2) conditions has a unique global-in-time classical solution.*

Background on linear systems

Consider the linear system

$$v_t + \sum_{i=1}^{n} A_i(x,t) v_{x_i} = h(x,t) \quad \text{in } U_T, \tag{3.1}$$

satisfying (C-1) with initial value $v(x,0) \doteq v(0) = v_0(x)$.

Proposition 3.2 *Suppose that*

$$v \in C([0,T], H^k) \cap C^1([0,T], H^{k-1}),$$

satisfies the initial value problem to (3.1), then u satisfies the energy estimate

$$\max_{0 \le t \le T} (\|v(t)\|_k + \|v_t(t)\|_{k-1}) \le C_k e^{\beta_k T} \left(\|v(0)\|_k + \int_0^T \|h(s)\|_k \, ds \right), \tag{3.2}$$

where the constants C_k and β_k depend on the H^k–norms of A_i.

Proposition 3.3 *The initial value problem to the linear system (3.1) has a unique solution of class $C([0,T], H^k) \cap C^1([0,T], H^{k-1})$.*

Proof of Theorem 3.1

The proof is given in three main steps. In step 1, we set up an approximate iteration of linear systems along with a corresponding transformation related to the global solutions to these linear systems. In step 2 we show that this transformation is a strict contraction on an appropriate function space for sufficiently small time. Further, the unique fixed point of this contraction map agrees with the unique short-time classical solution of the quasilinear IVP. Step 3 applies Sobolev energy estimates to extend this unique classical solution further in time.

Proof **Step 1:** Choose an arbitrary $T > 0$. We shall prove existence of solutions up to this arbitrary time. First construct the linear problem:

$$v_t + \overline{A_i(x,t,u(x,t))}v_{x_i} = h(x,t,u) \quad \text{in } U_T, \tag{3.3}$$

$$v(x,0) = u_0(x). \tag{3.4}$$

where the subscript i is short-hand for summation from 1 to n. The global existence and uniqueness of solutions $v \in C([0,T],H^k) \cap C([0,T],H^{k-1})$ to this IVP holds. The first step to showing existence of a solution of the quasilinear system is to consider the transformation $\mathcal{T}$ defined by $v = \mathcal{T}u$ where u is given and v is the solution of (3.3)–(3.4). Our goal is to prove this transformation is a strict contraction on a suitable function space. We consider

$$u \in X^{k,\tau} \doteq C([0,\tau],H^k(\mathbb{R}^n,\mathbb{R}^N)).$$

Using the energy estimates, one has

$$\max_{0\le t\le\tau} \left(\|v(t)\|_k + \|v_t(t)\|_{k-1}\right) \le C_k e^{\beta_k \tau}\left(\|v(0)\|_k + \int_0^\tau \|h(s)\|_k\, ds\right). \tag{3.5}$$

Define

$$B_R^{k,\tau} \doteq \{u \in X^{k,\tau} : \ \|u\|_{X^{k,\tau}} \le R\}.$$

It is clear from (3.5) that $\mathcal{T}$ maps $B_R^{k,\tau}$ to itself for sufficiently small τ and a suitable R. We now show that $\mathcal{T}$ is a contraction on $B_R^{k,\tau}$ in the $X^{0,\tau}$-norm. Let $v_j = \mathcal{T}u_j$ for $j = 1,2$ and set $w = v_1 - v_2$. Then w satisfies the linear system

$$w_t + A_i(u_1)w_x = H(x,t) \text{ and } w(0) = 0,$$

where $H(x,t) = h(x,t,u_1) - h(x,t,u_2) + (A_i(x,t,u_2) - A_i(x,t,u_1))(v_2)_x$. From the Lipschitz continuity of A_i and h with respect to u and the Sobolev embedding, $\|H(t)\|_0 \le C\|u_1 - u_2\|_0$ where the constant C depends on R and the Lipschitz constants of A_i and h. Using the energy estimate (3.2), we obtain

$$\max_{0\le t\le\tau} \|\mathcal{T}u_1 - \mathcal{T}u_2\|_0^2 \le C_0 e^{\beta_0\tau}\tau \max_{0\le t\le\tau} \|u_1 - u_2\|_0^2.$$

Hence $\mathcal{T} : B_R^{k,\tau} \mapsto B_R^{k,\tau}$ is a strict contraction with respect to the $X^{0,\tau}$-norm for sufficiently small τ.

Consider the iteration scheme: let $u^{(j+1)} = \mathcal{T}u^{(j)}$ with $u^{(0)} = u_0$. As a consequence of the contraction mapping principle, $u^{(j)}$ converges to a unique $u \in X^{0,\tau}$ i.e.

$$\lim_{j\to\infty} \max_{0\le t\le\tau} \|u^{(j)} - u\|_0 = 0. \tag{3.6}$$

Step 2: We show in this step that this limiting function u belongs in $C^1(\mathbb{R}^n \times [0,\tau], \mathbb{R}^N)$. Energy estimates and interpolation inequalities imply that, for any s with $0 \leq s < k$,

$$\begin{aligned} \max_{0\leq t\leq\tau} \|u^{(j)} - u^{(l)}\|_s &\leq C \max_{0\leq t\leq\tau} \|u^{(j)} - u^{(l)}\|_0^{1-s/k} \max_{0\leq t\leq\tau} \|u^{(j)} - u^{(l)}\|_k^{s/k} \\ &\leq C \max_{0\leq t\leq\tau} \|u^{(j)} - u^{(l)}\|_0^{1-s/k}. \end{aligned} \tag{3.7}$$

It follows from this and (3.6) that

$$\lim_{j\to\infty} \max_{0\leq t\leq\tau} \|u^{(j)} - u\|_s = 0,$$

for any $0 \leq s < k$. That is,

$$u^{(j)} \to u \in C([0,\tau], H^s(\mathbb{R}^n, \mathbb{R}^N)).$$

This along with the PDE itself yields $u \in C^1([0,\tau], H^{s-1}(\mathbb{R}^n, \mathbb{R}^N))$ provided that $s > 1 + n/2$. Thus

$$u \in C([0,\tau], H^s(\mathbb{R}^n, \mathbb{R}^N)) \cap C^1([0,\tau], H^{s-1}(\mathbb{R}^n, \mathbb{R}^N)),$$

i.e. $u = u(x,t)$ is a classical solution by the Sobolev embedding.
Step 3: In this step, we extend the local classical solution of the whole interval $[0,T]$ through energy estimates. Once this a priori estimate is established, it will allow us to repeat the above local existence argument on $(\tau, 2\tau)$, $(2\tau, 3\tau)$, $(3\tau, 4\tau), \ldots$ until we have covered $[0,T]$. Let us establish the following energy estimate.

Energy Estimate:
Suppose that $u \in C([0,\tau], H^k(\mathbb{R}^n, \mathbb{R}^N)) \cap C^1([0,\tau], H^{k-1}(\mathbb{R}^n, \mathbb{R}^N))$ satisfies the initial value problem (2.2), then u satisfies the energy estimate.

$$\max_{0\leq t\leq T} (\|v(t)\|_k + \|v_t(t)\|_{k-1}) \leq C_k(T). \tag{3.8}$$

Take the L^2 inner product between u and (2.2), integrate over space to get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2\,dx + \int_{\mathbb{R}^n} u \cdot \overline{A_i(x,t,u)} u_{x_i}\,dx = \int_{\mathbb{R}^n} u \cdot h(x,t,u)\,dx.$$

Using the symmetry of A and integration by parts, the second term on the left-hand side becomes

$$\int_{\mathbb{R}^n} u \cdot \overline{A_i(x,t,u)} u_{x_i}\,dx = -\frac{1}{2}\int_{\mathbb{R}^n} \partial_{x_i} \overline{A_i(x,t,u)} u \cdot u\,dx.$$

Then Young's inequality and the Sobolev embedding imply

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_0^2 &\leq \left(\|G\|_{1,1} \|A_i(x,t,u)\|_{L^\infty}\right) \|u(t)\|_0^2 + 2\|h(t)\|_0^2 \\ &\leq C\|u(t)\|_0^2, \end{aligned}$$

so Gronwall's inequality yields

$$\|u(t)\|_0^2 \leq e^{CT} \|u(0)\|_0^2.$$

To establish the estimate for higher-order derivatives, for $|\alpha| \leq k$ apply D^α to (2.2), product with $D^\alpha u$ and integrate over space.

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} |D^\alpha u|^2\, dx \\
&\quad = \int_{\mathbb{R}^n} D^\alpha u \cdot D^\alpha h \, dx - \int_{\mathbb{R}^n} D^\alpha u \cdot D^\alpha \left(\overline{A_i(x,t,u)} u_{x_i}\right) dx \\
&\quad = \int_{\mathbb{R}^n} D^\alpha u \cdot D^\alpha h \, dx \\
&\qquad - \left(\underbrace{\int_{\mathbb{R}^n} D^\alpha u \cdot \overline{A(x,t,u)}\,(D^\alpha u)_{x_i}\; dx + \ldots + \int_{\mathbb{R}^n} D^\alpha u \cdot D^\alpha \overline{A(x,t,u)} u_{x_i}\, dx}_{\doteq I} \right).
\end{aligned} \tag{3.9}$$

The first term in I is handled as in the L^2 case using symmetry and integration by parts,

$$\begin{aligned}
-\int_{\mathbb{R}^n} D^\alpha u \cdot \overline{A_i(x,t,u)}\,(D^\alpha u)_{x_i}\; dx &= \frac{1}{2}\int_{\mathbb{R}^n} \partial_{x_i} \overline{A(x,t,u)} D^\alpha u \cdot D^\alpha u \, dx \\
&\leq \frac{1}{2} \|G\|_{1,1} \|A_i(x,t,u)\|_{L^\infty} \int_{\mathbb{R}^n} |D^\alpha u|^2\, dx.
\end{aligned}$$

The last term in I is handled by the Sobolev embedding,

$$\begin{aligned}
\int_{\mathbb{R}^n} D^\alpha u \cdot D^\alpha \overline{A_i(x,t,u)} u_{x_i}\, dx &\leq \|u_{x_i}\|_{L^\infty} \|D^\alpha u\|_0 \|D^\alpha \overline{A_i(x,t,u)}\|_0 \\
&\leq C\|u\|_k^2.
\end{aligned}$$

The intermediate terms in I are handled similarly. These estimates for (3.9) yield

$$\frac{d}{dt}\|u(t)\|_k^2 \leq C\|u(t)\|_k^2,$$

so by Gronwall's inequality,

$$\|u(t)\|_k \leq C_k(T).$$

Similarly, we can use (2.2) directly with the last estimate to compute

$$\begin{aligned}
\|\partial_t u(t)\|_{k-1} &\leq \|h(x,t,u)\|_{k-1} + \|\overline{A_i(x,t,u)} u_{x_i}\|_{k-1} \\
&\leq C\|u(t)\|_{k-1} + \|\overline{A_i(x,t,u)}\|_{k-1} \|u(t)\|_k \\
&\leq C_k(T).
\end{aligned}$$

This completes the proof of the theorem.

4 Convergence to the admissible weak solution

Consider the following non-homogeneous scalar conservation law,

$$\partial_t u + f(u)_x = h(x,t,u) \quad \text{in } U_T = \mathbb{R} \times (0,T), \tag{4.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{4.2}$$

where u, f, h are real-valued functions on U_T. Let $a \doteq \dfrac{\partial f}{\partial u}$ and place the same conditions on the resulting averaged equation,

$$\partial_t u^\alpha + \overline{a(u^\alpha)} u_x^\alpha = h(x,t,u^\alpha) \text{ in } U_T, \tag{4.3}$$

$$u^\alpha(x,0) = u_0(x) \text{ for } x \in \mathbb{R}, \tag{4.4}$$

as was given in Theorem 3.1. In addition to these previous assumptions, we place further conditions as follows.

Further Assumptions: Let $h(x,t,0) \in C([0;T]; L^1(\mathbb{R}))$, the initial data u_0 is of class $H^k \cap W^{1,1}(\mathbb{R})$ (so it has bounded total variation), and let the flux f and the filter G satisfy the following property:
For each $\alpha \ll 1$ and for the corresponding regularized solutions, v^α, of (4.3)–(4.4),

$$\partial_x a(v^\alpha) \geq \partial_x \overline{a(v^\alpha)}. \tag{4.5}$$

Remark: From the previous section, there is no gradient blowup for each $\alpha > 0$, so the spatial derivatives of the coefficients in (4.5) exist in the classical sense. Hence, we may interpret this condition as having a low-pass filter that exacts control as it acts on the gradient by bounding it below by its mollified version. This suggests that the chosen averaging kernel may play a role in establishing condition (4.5) in addition to the flux and the initial data. This observation is revisited for the Helmholtz filter shortly below.

Our aim for this section is to verify that the global classical solutions for the Cauchy problem to (4.3)-(4.4) does in fact converge to a weak solution for the Cauchy problem to (4.1)–(4.2) as $\alpha \longrightarrow 0^+$. By a weak solution we mean a solution of the Cauchy problem in the following sense.

Definition 4.1 A function $u : \mathbb{R} \times [0,T] \mapsto \mathbb{R}^N$ is a weak solution of the Cauchy problem (4.1)–(4.2) if u is continuous as a function from $[0,T]$ into L^1_{loc}, the initial condition (4.2) holds and the restriction of u to the open strip U_T is a distributional solution i.e.

$$\int_0^T \int_{-\infty}^{\infty} u\phi_t + f(u)\phi_x + h(x,t,u)\phi \, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\, dx = 0, \tag{4.6}$$

for every C^∞ function ϕ with compact support contained in the set $\mathbb{R} \times (-\infty, T)$.

The notion for proving the convergence result is summarized in two key steps. In step 1, the needed uniform, BV, and L^1 estimates are established on the sequence of averaged solutions $\{u^\alpha\}_{\alpha>0}$ that guarantee compactness in $C([0,T), L^1_{loc}(\mathbb{R}))$. In step 2 the limit function in the $\alpha \to 0^+$ limit is shown to satisfy the definition of a weak solution for the Cauchy problem. For the homogeneous case, this weak solution is, in fact, entropy admissible.

Theorem 4.2 *The sequence of averaged solutions $u^\alpha : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$, indexed by $\alpha > 0$, of (4.3)–(4.4) has a subsequence $\{u^\beta\}_{\beta>0}$ that converges strongly to some function u in $C([0,T], L^1_{loc}(\mathbb{R}))$ as $\beta \to 0^+$. Moreover, this limit u is a weak solution of (4.1)–(4.2), and we refer to this as the α–limit weak solution hereafter.*

Proof **Step 1:** In order show the convergence of the solution sequence to a weak solution, an L^1–compactness result is invoked. In order to do so, we first establish the following estimates:

(i) $|u^\alpha(x,t)| \leq M_1$ for all x, t,

(ii) $T.V.(u^\alpha(\cdot,t)) \le M_2$ for all t,
(iii) $\int_{\mathbb{R}} |u^\alpha(x,t+s) - u^\alpha(x,t)|\,dx \le M_3 s$ for all $s,t \ge 0$,

where the constants M_i for $i = 1,2,3$ are independent of α. These estimates imply that there exists a subsequence (indexed by α still) and a function u such that $\lim_{\alpha\to 0} u^\alpha = u$ in $C([0,T], L^1_{loc}(\mathbb{R}))$ [4].
To show estimate (i), the uniform boundedness of u^α, recall that u^α satisfies the system in the mild[22] or broad[4] sense i.e.

$$u^\alpha(\xi,\tau) = u_0(x(0;\xi,\tau)) + \int_0^\tau h(x(t;\xi,\tau),t,u^\alpha(x(t;\xi,\tau),t)\,dt,$$

where $t \mapsto x(t;\xi,\tau)$ is the unique integral solution of the IVP

$$\begin{aligned} \dot{x} &= \overline{a(u^\alpha(x,t))}, \\ x(\tau) &= \xi. \end{aligned}$$

So we have the following,

$$\begin{aligned} |u^\alpha(\xi,\tau)| &\le \|u_0\|_{L^\infty} + \int_0^\tau |h(x(t;\xi,\tau),t,u^\alpha(x(t;\xi,\tau),t)|\,dt \\ &\le \|u_0\|_{L^\infty} + C_0 + |h|_{\mathrm{Lip}} \sup_{(\xi,\tau)} \int_0^\tau |u^\alpha(x(t;\xi,\tau),t)|\,dt, \end{aligned}$$

where $|h(x,t,0)|_{L^\infty} \le C_0$. By Gronwall's inequality

$$\sup_{(\xi,\tau)} |u^\alpha(\xi,\tau)| \le (\|u_0\|_{L^\infty} + C_0)e^{|h|_{\mathrm{Lip}}T} =: M_1.$$

To show estimate (ii), the uniform boundedness of the total variation of the solutions independent of α, differentiate (4.3) with respect to the spatial variable x to obtain

$$\partial_t u^\alpha_x + (\overline{a(u^\alpha)} u^\alpha_x)_x = \partial_x(h(x,t,u)). \tag{4.7}$$

Multiply (4.7) by the sign function $sgn(u^\alpha_x)$ then integrate over $\mathbb{R}$ with respect to x,

$$\frac{d}{dt}\int_{\mathbb{R}} |u^\alpha_x|\,dx + \int_{\mathbb{R}} (\overline{a(u^\alpha)} u^\alpha_x)_x sgn(u^\alpha_x)\,dx = \int_{\mathbb{R}} \partial_x(h(x,t,u^\alpha)) sgn(u^\alpha_x)\,dx. \tag{4.8}$$

Observe that the second integral term on the left-hand side of (4.8) is zero by the continuity of both $\overline{a(u^\alpha)}$ and u^α_x. The right-hand side can be bounded above as follows,

$$\begin{aligned} \int_{\mathbb{R}} sgn(u^\alpha_x)\partial_x h(x,t,u^\alpha)\,dx &\le \int_{\mathbb{R}} \left|\frac{\partial h}{\partial x}\right| dx + \int_{\mathbb{R}} \left|\frac{\partial h}{\partial u}\right| |u^\alpha_x|\,dx \\ &\le C_1 + C_2 \int_{\mathbb{R}} |u^\alpha_x|\,dx. \end{aligned}$$

Thus $\frac{d}{dt}\|u^\alpha_x(\cdot,t)\|_{L^1} \le C_1 + C_2\|u^\alpha_x(\cdot,t)\|_{L^1}$ which implies

$$T.V.(u^\alpha(\cdot,t)) = \|u^\alpha_x(\cdot,t)\|_{L^1} \le (\|u_0'\|_{L^1} + C_1 T)\,e^{C_2 T} = C_T T.V.(u_0) =: M_2.$$

To obtain estimate (iii), observe that $\partial_t u^\alpha = -\overline{a(u^\alpha)}u_x + h(x,t,u^\alpha)$. Integrating this from the interval $[t, t+s]$ then integrating over the real line with respect to the x variable yields

$$\begin{aligned}
&\int_{\mathbb{R}} |u^\alpha(x,t+s) - u^\alpha(x,t)|\,dx \\
&\leq \int_{\mathbb{R}} \int_t^{t+s} |h(x,s,u^\alpha(x,r)) - \overline{a(u^\alpha(x,r))}u_x^\alpha(x,r)|\,dr\,dx \\
&\leq (C_1 + C_2 M_2)s.
\end{aligned}$$

for some constants C_1, C_2 independent of α.

Step 2: We claim that this limit function u is a weak solution.
Claim: The unique solutions to (4.3)–(4.4) satisfy

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} |u^\alpha - k|\phi_t + p(u^\alpha,k)\phi_x + sgn(u^\alpha - k)h(x,t,u^\alpha)\phi\,dxdt \\
&\qquad + \int_{\mathbb{R}} |u_0(x) - k|\phi(x,0)\,dx \\
&\quad = \int_0^T \int_{\mathbb{R}} |u^\alpha - k| \left(a(u^\alpha) - \overline{a(u^\alpha)}\right)\phi_x\,dx\,dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} |u^\alpha - k| \left(a(u^\alpha) - \overline{a(u^\alpha)}\right)_x \phi\,dx\,dt. \qquad (4.9)
\end{aligned}$$

for every $k \in \mathbb{R}$ and every non-negative test function $\phi \in C_c^\infty(\mathbb{R} \times (-\infty, T))$. Here $p(u,k) = sgn(u-k)(f(u) - f(k))$.
To see this, notice that the solutions to (4.3)–(4.4) will satisfy

$$\partial_t(u^\alpha - k) - (f(u^\alpha) - f(k))_x + h(x,t,u^\alpha) = (a(u^\alpha) - \overline{a(u^\alpha)})(u^\alpha - k)_x.$$

Multiply this equation by $sgn_\nu(u^\alpha - k)\phi$, where

$$sgn_\nu(x) = \begin{cases} -1, & \text{if } x < -\nu, \\ x/\nu, & \text{if } |x| \leq \nu, \\ 1, & \text{if } x > \nu, \end{cases}$$

then integrate over $U_T = \mathbb{R} \times (0,T)$. Let $w^\alpha := u^\alpha - k$ then integration by parts leads to the following identity,

$$\begin{aligned}
&\iint_{U_T} sgn_\nu(w^\alpha)(w^\alpha)\phi_t + sgn_\nu(w^\alpha)(f(u^\alpha) - f(k))\phi_x\,dxdt \\
&\qquad + \iint_{U_T} sgn'_\nu(w^\alpha) \cdot \phi[w^\alpha \cdot w_t^\alpha + w_x^\alpha(f(u^\alpha) - f(k))]\,dxdt \\
&\qquad + \iint_{U_T} sgn_\nu(w^\alpha)h(x,t,u^\alpha)\phi\,dxdt \\
&\qquad + \int_{\mathbb{R}} sgn_\nu(u_0(x) - k)(u_0(x) - k)\phi(x,0)\,dx \\
&\quad = \iint_{U_T} sgn_\nu(w^\alpha)\left(a(u^\alpha) - \overline{a(u^\alpha)}\right) w^\alpha \phi_x\,dxdt \\
&\qquad + \iint_{U_T} sgn'_\nu(w^\alpha)w_x^\alpha \left(a(u^\alpha) - \overline{a(u^\alpha)}\right) w^\alpha \phi\,dxdt \\
&\qquad + \iint_{U_T} sgn_\nu(w^\alpha)\left(a(u^\alpha) - \overline{a(u^\alpha)}\right)_x w^\alpha \phi\,dxdt.
\end{aligned}$$

By taking the limit as $\nu \longrightarrow 0$ and noting that the limits of the second integrals on both the left-hand and right-hand sides of this identity both converge to zero, we obtain (4.9). If $k = -\sup_{\alpha>0} \|u^\alpha\|_{L^\infty}$ then the absolute values and signs drop accordingly, and (4.5) and (4.9) imply that as $\alpha \to 0^+$,

$$\int_0^T \int_{-\infty}^{\infty} u\phi_t + f(u)\phi_x + h(x,t,u)\phi \, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\, dx \geq 0.$$

If $k = \sup_{\alpha>0} \|u^\alpha\|_{L^\infty}$, then the sign changes (≤ 0). Thus,

$$\int_0^T \int_{-\infty}^{\infty} u\phi_t + f(u)\phi_x + h(x,t,u)\phi \, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\, dx = 0.$$

This completes the proof of the theorem.

Now consider (4.1)–(4.2) in the absence of source terms:

$$\partial_t u + f(u)_x = 0 \quad \text{in } U_T, \tag{4.10}$$
$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \tag{4.11}$$

The following corollary shows that the sequence of solutions of the corresponding regularized problem,

$$\partial_t u^\alpha + \overline{a(u^\alpha)} u^\alpha_x = 0 \quad \text{in } U_T, \tag{4.12}$$
$$u^\alpha(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{4.13}$$

converges to the entropy admissible solution of (4.10)–(4.11).

Corollary 4.3 (homogeneous) *The α–limit weak solution of (4.12)–(4.13) is the unique entropy admissible solution of (4.10)–(4.11).*

Proof Let $v = v(x,t)$ be the unique Krŭzkov entropy admissible solution to (4.10)-(4.11). That is, $v = v(x,t)$ is a weak solution that satisfies the following entropy inequality: for every constant $k \in \mathbb{R}$ and every non-negative test function ϕ with compact support in $\mathbb{R} \times (0,T)$,

$$\int_0^T \int_{\mathbb{R}} |v-k|\phi_t + p(v,k)\phi_x \, dxdt \geq 0, \tag{4.14}$$

where $p(v,k) := sgn(v-k)(f(v)-f(k))$.

Similar to the previous calculations, we can show that the unique solution to (4.12) satisfies

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |u^\alpha - k|\phi_t + p(u^\alpha,k)\phi_x \, dxdt \\ = \int_0^T \int_{\mathbb{R}} |u^\alpha - k| \left(a(u^\alpha) - \overline{a(u^\alpha)} \right) \phi_x \, dx\, dt \\ + \int_0^T \int_{\mathbb{R}} |u^\alpha - k| \left(a(u^\alpha) - \overline{a(u^\alpha)} \right)_x \phi \, dx\, dt\end{aligned} \tag{4.15}$$

for every $k \in \mathbb{R}$ and every non-negative test function ϕ. Now by compactness and (4.5), taking the α–limit to (4.15) shows that the α–limit weak solution u is Krŭzkov entropy admissible. Specifically, the first integral on the right-hand side of (4.15) will limit to zero by compactness and the second integral on that same side is non-negative. By the L^1–stability of entropy solutions,

$$\int_{\mathbb{R}} |u(x,t)-v(x,t)|\,dx \le \int_{\mathbb{R}} |u(x,0)-v(x,0)|\,dx = 0.$$

That is, the sequence of regularized solutions $\{u^\alpha\}$ must converge to u—the unique entropy admissible solution to the IVP.

Remarks on the Helmholtz filter and quadratic fluxes

In order to build a better understanding of the conditions considered in Theorem 2 and its Corollary, we provide a motivating example which satisfy such conditions. As we shall see, the flux, filter, and initial data can play key roles in obtaining the conditions guaranteeing the convergence to the entropy admissible solution.

Let G be the Helmholtz filter and $f = f(u)$ be a second-order real polynomial with respect to u, $f(u) = c_0 + c_1 u + c_2 u^2$. This section shows that we can exploit the structure of the filter and the convex flux to prove convergence to the weak solution. For instance, we obtain the following

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} u^\alpha \phi_t + f(u^\alpha)\phi_x \,dxdt + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,dx \\
&= -\int_0^T \int_{\mathbb{R}} \left(a(u^\alpha) - \overline{a(u^\alpha)}\right) u^\alpha_x \phi \,dx\,dt \\
&= -\int_0^T \int_{\mathbb{R}} 2c_2(u^\alpha - \overline{u^\alpha}) u^\alpha_x \phi \,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}} 2c_2(\alpha^2 \overline{u^\alpha}_{xx})(\overline{u^\alpha}_x - \alpha^2 \overline{u^\alpha}_{xxx})\phi \,dx\,dt \\
&= \alpha^2 c_2 \int_0^T \int_{\mathbb{R}} \partial_x \left(\overline{u^\alpha}_x^2 - \alpha^2 \overline{u^\alpha}_{xx}^2\right) \phi \,dx\,dt \\
&= c_2 \int_0^T \int_{\mathbb{R}} \left(\alpha^2 \overline{u^\alpha}_{xx}\right)^2 \phi_x \,dx\,dt - \alpha^2 c_2 \int_0^T \int_{\mathbb{R}} \left(\overline{u^\alpha}_x\right)^2 \phi_x \,dx\,dt \\
&:= E_1 + E_2.
\end{aligned}$$

E_1 must limit to zero as $\alpha \to 0^+$ from the estimate

$$\begin{aligned}
c_2 \int_0^T \int_{\mathbb{R}} \left|\alpha^2 \overline{u^\alpha}_{xx}\right|^2 \phi_x \,dx\,dt &\le c_2 \int_0^T \int_{\mathbb{R}} \left|\overline{u^\alpha} - u^\alpha\right|^2 |\phi_x| \,dx\,dt \\
c_2 &\le \underbrace{\|\overline{u^\alpha} - u^\alpha\|_{L^\infty}}_{\le 2\|u_0\|_{L^\infty}} \|\phi_x\|_{L^\infty} \int_{supp(\phi)} \left|\overline{u^\alpha} - u^\alpha\right| \,dx\,dt \\
&\le 2c_2 M_1 \|\phi_x\|_{L^\infty} \int_{supp(\phi)} \left|\overline{u^\alpha} - u^\alpha\right| \,dx\,dt \longrightarrow 0 \text{ as } \alpha \longrightarrow 0^+.
\end{aligned}$$

It remains to be shown that the term E_2 limits to zero as $\alpha \to 0^+$. This follows from the following estimate

$$\begin{aligned}
|E_2| &\leq \alpha^2 c_2 \int_0^T \int_{\mathbb{R}} \left| \left(\overline{u^\alpha}_x\right)^2 \phi_x \right| \, dx\, dt \\
&\leq \alpha^2 c_2 \underbrace{\|\overline{u^\alpha}_x\|_{L^\infty}}_{\leq \alpha^{-1}\|u_0\|_{L^\infty}} \|\phi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} |\overline{u^\alpha}_x| \, dx\, dt \\
&\leq \alpha c_2 M_1 \|\phi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} |\overline{u^\alpha}_x| \, dx\, dt \\
&\leq \alpha c_2 M_1 M_2 \|\phi_x\|_{L^\infty} T \longrightarrow 0 \text{ as } \alpha \longrightarrow 0^+.
\end{aligned}$$

Moreover, to apply Corollary 1, we impose the condition that for $\alpha \ll 1$, $u_x^\alpha \geq \overline{u^\alpha}_x$ or $-\alpha^2 \overline{u^\alpha}_{xxx} = -\alpha^2 g_x * u_{xx}^\alpha \geq 0$ by definition of the Helmholtz filter. One can certainly choose initial conditions and exploit the properties of the filter so that such a condition holds—similar to what was done in [28]. Interestingly, these estimates illustrate why many successful results for this regularization were seen for the Burgers' and homentropic Euler equations under the Helmholtz filter.

5 Concluding Remarks

The results provided here validates our method as a successful shock-regularization of scalar conservation laws including the basic inviscid Burgers' equation (1.1) proposed in [24]. Moreover, the global existence result in section 3 includes the higher-dimensional Burgers' equation studied in [27]:

$$\begin{cases} \partial_t u + (\overline{u} \cdot \nabla) u = 0 & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{5.1}$$

since this is equivalent to (2.2) with

$$A_i(x,t,u) \doteq \operatorname{diag}(u_i, u_i, \ldots, u_i).$$

This global existence result also includes the regularized 1d homentropic Euler equations introduced in [29]. Here the authors examined

$$\begin{cases} \rho_t + (\rho v)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ (\rho v)_t + (\rho u u + P)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \end{cases}$$

supplemented with initial data

$$(\rho, v)(x,0) = (\rho_0, v_0)(x),$$

where $P = \frac{\rho^\gamma}{\gamma}$ with $\gamma > 1$. Recall that the eigenvalues for this 2×2 system are

$$\lambda_1 = v - c, \qquad \lambda_2 = v + c,$$

where $c = \rho^\theta$, with $\theta = \frac{\gamma - 1}{2}$, is the sound speed. The corresponding Riemann-invariants,

$$w_1 = w_1(\rho, v) = v + \frac{\rho^\theta}{\theta}, \qquad w_2 = w_2(\rho, v) = v - \frac{\rho^\theta}{\theta},$$

provide a coordinate transformation that diagonalizes the system:

$$\begin{cases} \partial_t w_1 + \lambda_2(w_1, w_2) \partial_x w_1 = 0, \\ \partial_t w_2 + \lambda_1(w_1, w_2) \partial_x w_2 = 0. \end{cases}$$

The authors regularized this diagonalized system exactly as in our procedure by replacing the coefficients λ_i with the averaged quantity $\overline{\lambda}_i$. This averaging of the diagonalized system, unfortunately, does not capture the entropy solution to the homentropic Euler equations as was achieved for scalar equations. We include this example because it illustrates the difficulty in successfully extending our technique to other systems even those which are not initially symmetric although symmetrizable. Recently, however, there have been several proposed extensions to our filtering method. For instance, the authors in [23, 26] introduced the observable divergence method as a natural extension of the filtering in the inviscid Burgers' equation to both the homentropic Euler and full Euler equations in higher dimensions. They numerically studied these regularized models and it did appear to add regularity to solutions while capturing the entropy solution for the shock-tube and the Shu-Osher problems. In addition, we placed further assumptions on the filtered conservation laws in order to show that the limiting solution was entropy admissible. As demonstrated earlier, these assumptions resulted from the estimates we have obtained. Therefore, another aspect for future examination will concern developing sharper estimates, if possible, that will allow us to weaken these assumptions. Further, one can consider spatial averaging applied to initial boundary value problems to conservation laws. Initial-boundary value problems for conservation laws have been studied in [1, 2, 9, 17]. Due to the boundary effects, entropy inequalities can be formulated using either vanishing viscosity or the Riemann problem just as what is done for the purely initial value problem. A natural question is if our inviscid regularization can be applied in this setting in the sense that this method recovers the boundary entropy inequalities.

Nonetheless, it is our hope that, perhaps, analogous results for these generalized techniques can be developed as was done here for scalar conservation laws and, at the very least, encourage the study, development, and consideration of using similar filtering techniques in the simulation of the models for inviscid flows.

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